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Courtlandt L. Bohn

Fermi National Accelerator Laboratory
P.O. Box 500, Batavia, Illinois 60510

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Fermilab, Batavia, IL 60510-0500

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Abstract

The e-folding time by which an initially localized perturbation disperses throughout a charged-particle bunch characterizes the irreversible process of chaotic mixing. In the presence of stationary, nonlinear space-charge forces, this time scale is estimated to be a few space-charge-depressed betatron periods, much shorter than the collisional relaxation time. The estimate derives from a geometrodynamic technique that is applied to quantify the exponential separation of nearby trajectories and their eventual distribution over a microcanonical ensemble.

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Rapid irreversible dynamics is a central concern in producing high-brightness particle beams. Time scales of irreversible processes place constraints on methods for compensating against degradation of beam quality. Examples include reversing deleterious effects from space charge in high-brightness injectors [1], and from coherent synchrotron radiation in accelerators that power modern free-electron lasers [2]. Compensation needs to be done on a time scale fast compared to the irreversible process, which in turn affects the choice and configuration of the associated hardware.

The past few years have seen development of a geometrodynamic technique for analytically estimating the largest Lyapunov exponent in a system having many degrees of freedom [3]. It concerns mixing of chaotic orbits through the configuration space on an exponential time scale. The mixing is irreversible in the sense that infinitesimally small fine-tuning is needed to reassemble the initial conditions. It is also distinctly different from phase mixing (linear Landau damping), a regular, reversible process that "winds up" the phase space through a distribution of orbital frequencies. Phase mixing proceeds on a relatively slow time scale set by the distribution of frequencies.

Space charge will typically result in a nonintegrable potential, *i.e.*, one in which the degrees of freedom are coupled. Systems having at least two degrees of freedom and a nonintegrable potential can be expected to comprise a significant percentage of chaotic orbits. This is true even if the potential is stationary, the case under consideration in what follows.

A charged-particle bunch will generally have a density distribution with a Debye tail [4]. Interior particles that never reach into the Debye tail are effectively screened from external forces; however, particles reaching into the Debye tail will respond to these forces and communicate their influence throughout the bunch. Nonlinear space-charge forces likewise predominate in the Debye tail, so these same particles are also the ones that are most susceptible to chaotic behavior. Moreover, on a local scale, a bunch is fully $3N$-dimensional, with N representing the number of particles. All of the constituent particles are therefore under the influence of space-charge fluctuations, and conditions can be such that a generic orbit is chaotic.

We now turn toward calculating the time scale for mixing. Action principles in classical mechanics are tantamount to extremals of "arc lengths;" thus, one can infer a metric tensor from an action principle [5]. The metric tensor manifests all of the properties of the manifold over which the system evolves, with these properties being calculable following standard principles of differential geometry [6]. Of special interest for determining Lyapunov exponents, quantities that measure the exponential rate at which initially localized trajectories separate, is the equation of geodesic deviation:

$$\frac{D^2\delta q^\alpha}{ds^2} + R^\alpha{}_{\beta\gamma\delta}\frac{dq^\beta}{ds}\delta q^\gamma \frac{dq^\delta}{ds} \;=\; 0, \tag{1}$$

in which $\mathbf{q}(s)$ denotes the coordinate vector of the system, $\delta\mathbf{q}(s)$ represents the separation vector between neighboring geodesics at "proper time" s, D/ds denotes covariant differentiation, $R^\alpha{}_{\beta\gamma\delta}$ is the Riemann tensor derivable from the metric tensor, and summation over repeated indices is implied with each index spanning the number of degrees of freedom accessible to the system. Equation (1) is fundamental for determining a Lyapunov exponent λ because it is defined in terms of the separation vector:

$$\lambda \;=\; \lim_{t\to\infty}\frac{1}{t}\ln\frac{|\delta\mathbf{q}(t)|}{|\delta\mathbf{q}(0)|}. \tag{2}$$

Any number of action principles, and therefore any number of metric tensors, can be selected to proceed further. Eisenhart's metric [7], which is consistent with Hamilton's least-action principle, is probably the most convenient choice. It offers easy calculation of the Riemann tensor, and it avoids spurious results traceable to the singular boundary of the perhaps better-known Jacobi metric that is derivable from Maupertius' least-action principle [8]. Eisenhart's metric operates over a combined space-time manifold in which the geodesics are parameterized by the real time t, *i.e.*, $ds^2 = dt^2$ with

$$ds^2 \;=\; -\,2V(\mathbf{q})(dq^0)^2 + \delta_{ij}dq^i dq^j + 2dq^0 dq^{\mathcal{N}+1}, \tag{3}$$

in which $V(\mathbf{q})$ is the potential, δ_{ij} is the unit tensor corresponding (without loss of generality) to a cartesian spatial coordinate system, the indices i, j run from 1 to the number of degrees of freedom $\mathcal{N}$ $(= 3N$ generally), $q^0 = t$, $q^{\mathcal{N}+1} = t/2 - \int_0^t dt' L(\mathbf{q}, \dot{\mathbf{q}})$, and L is the Lagrangian. The resulting geodesic equations for the spatial coordinates q^i are Hamilton's equations of motion, so the particle trajectories correspond to a canonical projection of the Eisenhart geodesics onto the configuration's space-time manifold. A convenient byproduct of the Eisenhart metric is that the only nonzero components of the Riemann tensor are $R_{0i0j} = \partial_i \partial_j V$, in which $\partial_i = \partial/\partial q^i$. In turn, the only nonzero component of the Ricci tensor $R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\gamma\beta}$ is $R_{00} = \partial^i \partial_i V$.

Rather than evaluate the geodesic deviation along all possible directions, we instead construct and study a generic geodesic deviation. To set up the governing equation, we put $\delta\mathbf{q} \propto \sqrt{\psi}\hat{\mathbf{n}}$, with $\hat{\mathbf{n}} = \delta\mathbf{q}/|\delta q|$ denoting the unit vector. Then, recognizing that there is no exponential growth of a deviation parallel to the geodesic, we average Eq. (1) over all $\mathcal{N} - 1$ perturbations $\delta\mathbf{q}$ that are orthogonal to the reference geodesic. The resulting equation of "generic" geodesic deviation is [9]

$$\frac{d^2\psi}{dt^2} + \frac{\mathcal{H}^{\beta}{}_{\beta}}{\mathcal{N} - 1}\psi \;=\; 0, \tag{4}$$

in which $\mathcal{H}^{\beta}{}_{\beta} = R_{\mu\nu}\dot{q}^{\mu}\dot{q}^{\nu} = \partial^i \partial_i V$ measures the curvature of the manifold along the geodesics, *i.e.*, the particle trajectories. Although the potential $V(\mathbf{q})$ is taken to be explicitly time-independent, an orbiting particle nevertheless sees a time-dependent curvature given that it is measured with reference to the particle's geodesic parameterized by the time t. The potential is a function of the coordinate vector $\mathbf{q}$, which in turn is implicitly time-dependent.

While orbiting in the self-consistent potential, each particle responds to the "bumpiness," *i.e.*, the local curvature, it sees, the bumpiness being associated with parametric resonances between the instantaneous orbital period and the effective nonlinear forces. For a chaotic orbit the resonances may be regarded to act stochastically, and in turn the equation of generic geodesic deviation may be regarded to be that of a stochastic oscillator,

$$\frac{d^2\psi}{dt^2} + [\kappa_t + \sigma_t \eta(t)]\,\psi \;=\; 0, \tag{5}$$

in which κ_t and σ_t are related to the time-averaged curvature and fluctuations, respectively:

$$\kappa_t \;=\; \frac{\langle \mathcal{H}^{\beta}{}_{\beta}\rangle_t}{\mathcal{N}-1}, \quad \sigma_t = \frac{\sqrt{\left\langle (\mathcal{H}^{\beta}{}_{\beta})^2 - \langle \mathcal{H}^{\beta}{}_{\beta}\rangle_t^2 \right\rangle_t}}{\sqrt{\mathcal{N}-1}}, \tag{6}$$

and $\eta(t)$ denotes a gaussian stochastic process. Specifically, $\langle \eta(t)\eta(t-\tau)\rangle = \tau\delta(t)$, where the average is taken over all realizations of the process, and τ is the correlation time, which is intermediate between the time required for the particle to traverse the average curvature radius and the time for it to transit the length scale of a typical fluctuation [3]. A gaussian process is the zeroth-order approximation of a cumulant expansion of the actual stochastic process.

As $t \to \infty$, the limit of interest for calculating a Lyapunov exponent, orbits of total energy E that mix through the configuration space will evolve toward an invariant measure, specifically the microcanonical ensemble $\mu = \delta(H-E)$, over which time averages become equivalent to phase-space averages. Anticipating that space charge will typically establish a preponderance of chaotic orbits, we consider it reasonable to average over the microcanonical ensemble, which is tantamount to invoking the so-called "chaotic hypothesis" [10]. Accordingly, for an arbitrary function $A(\mathbf{q})$ of the spatial coordinate $\mathbf{q}$, the average becomes

$$\lim_{t\to\infty}\langle A\rangle_t \;=\; \langle A\rangle_\mu \;=\; \frac{\int d\mathbf{q}\int d\dot{\mathbf{q}}\; A(\mathbf{q})\delta[H(\mathbf{q},\dot{\mathbf{q}})-E]}{\int d\mathbf{q}\int d\dot{\mathbf{q}}\; \delta[H(\mathbf{q},\dot{\mathbf{q}})-E]}, \tag{7}$$

and the generic geodesic approximately adheres to the simplified stochastic-oscillator equation

$$\frac{d^2\psi}{dt^2} + [\kappa + \sigma\eta(t)]\,\psi \;=\; 0, \tag{8}$$

in which κ, σ is shorthand notation for $\langle\kappa\rangle_\mu$, $\langle\sigma\rangle_\mu$, respectively. Note that all averages $\langle A\rangle_\mu$ are functions of the total particle energy.

Van Kampen [11] calculates from Eq. (8) the evolution of the process-averaged second moments of ψ, out of which the Lyapunov exponent corresponding to the mixing rate appears as a function of the correlation time τ [3]. Using a reasonable estimate of τ, one can write the Lyapunov exponent in the following convenient analytical form:

$$\lambda(\rho) = \frac{1}{\sqrt{3}}\frac{L^2(\rho)-1}{L(\rho)}\sqrt{\kappa};$$
$$L(\rho) = \left[T(\rho)+\sqrt{1+T^2(\rho)}\right]^{1/3}, \quad T(\rho) = \frac{3\pi\sqrt{3}}{8}\frac{\rho^2}{2\sqrt{1+\rho}+\pi\rho}; \tag{9}$$

in which $\rho \equiv \sigma/\kappa$, a quantity that measures the ratio of the average curvature radius to the length scale of fluctuations [12]. A plot of $\lambda/\sqrt{\kappa}$ versus ρ is given in Fig. 1; it shows that the time scale for mixing, $1/\lambda$, is a ρ-dependent multiple of $\kappa^{-1/2}$, so that $\kappa^{-1/2}$ is the fundamental time scale governing the irreversible process. Near the origin the curve scales as ρ^2, and at large ρ it scales as $\rho^{1/3}$.

Obtaining Eq. (9) seems to involve many approximations; however, they are all summarized concisely by the notion that a generic trajectory is chaotic, governed by a gaussian random process under the influence of parametric resonances, and evolving toward the microcanonical ensemble, an invariant measure. Numerical experiments, principally concerning condensed-matter and stellar systems, historically guided the reasoning that leads to the estimate. For example, Ref. [3] summarizes simulations of coupled-spin systems with long-range interactions and shows the estimated Lyapunov exponents agree remarkably well with computed values.

Because our interest is rapid mixing, we regard space charge to influence a given particle primarily through a coarse-grained potential that is everywhere proportional to the local particle density. Discrete collisions are presumed to alter the dynamics over a relaxation time much longer than the mixing time. A coarse-grained potential by no means vitiates the mechanism of parametric resonance. On the contrary, a Fourier frequency spectrum of a self-consistent potential associated with nonuniform density will generally feature continua. Consequently, ample opportunities will present themselves for parametric resonances between the fundamental frequency of a generic particle orbit and the field through which it orbits.

The foregoing results are now applied to a single charge orbiting in a smooth, three-dimensional ($\mathcal{N}=3$) potential $V = V_o + V_s$, with V_o denoting the potential associated with focusing forces external to the bunch, and V_s denoting the coarse-grained potential associated with space-charge forces internal to the bunch. For concreteness we take the external forces to be linear, so V_o is quadratic in the coordinates, *i.e.*, $V_o(\mathbf{q}) = (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2)/2$. Application of Poisson's equation then gives a simple result:

$$\nabla^2 V \;=\; \omega_o^2 - \frac{e}{\epsilon_o} n(\mathbf{q}), \tag{10}$$

in which $\omega_o^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$, e is the single-particle charge, ϵ_o is the permittivity of free space, and $n(\mathbf{q})$ is the (smoothed) particle density.

The quantities κ and σ are both determined from the Laplacian of the coarse-grained potential V_s. The results may be expressed conveniently in terms of the plasma frequency at the bunch centroid, ω_{po}, the space-charge-depressed focusing strength $\omega_s^2 = \omega_o^2 - \omega_{po}^2$, and the normalized particle density $\nu(\mathbf{q}) = n(\mathbf{q})/n(0)$:

$$\begin{aligned}
\kappa &= \frac{\omega_{po}^2}{2}\left[\left(\frac{\omega_s}{\omega_{po}}\right)^2 + 1 - \langle\nu\rangle\right], \\
\sigma &= \frac{\omega_{po}^2}{\sqrt{2}}\sqrt{\langle\nu^2\rangle - \langle\nu\rangle^2}, \\
\rho &= \frac{\sigma}{\kappa} \;=\; \frac{\sqrt{2}\sqrt{\langle\nu^2\rangle - \langle\nu\rangle^2}}{(\omega_s/\omega_{po})^2 + 1 - \langle\nu\rangle}.
\end{aligned} \tag{11}$$

From the estimated Lyapunov exponent as given by Eq. (9), one sees that the time scale for irreversible chaotic mixing, $\tau_m \equiv 1/\lambda$, is principally determined by the value of $1/\sqrt{\kappa}$ and is modified in the prescribed way by the value of ρ which carries the influence of the operative parametric resonances.

In realistic inhomogeneous density distributions, the value of ρ will typically be small but appreciable, and in turn Fig. 1 and Eq. (11) make clear that τ_m will typically be a few space-charge-depressed betatron periods. This agrees with, for example, the University of Maryland multiple-beam experiment showing distortion of an initial 5-beamlet configuration on time scales commensurate with a depressed betatron period [13]. Thus, to ensure mixing does not spoil a process of emittance compensation, a reasonable conservative criterion is that the process be completed within a plasma period, a criterion that translates into permissible beamline locations and maximum length that the associated hardware can occupy [14].

The mixing time is very short compared to the collisional relaxation time. The latter scales roughly as N_D/ω_p, where N_D is the number of particles in a Debye sphere, and is therefore typically several orders of magnitude longer than the mixing time. The mixing time calculated here also differs conceptually from a relaxation time. Geometrodynamics based on Eisenhart's metric takes no account of the evolution of velocity space. Moreover, a coarse-grained potential washes out the influence of collisions. Although we specialized to beams with space charge, we may nonetheless infer that, for a system in which parametric resonances cause typical particle trajectories to be chaotic, the mixing time will be governed principally by a global time scale derivable from the Lagrangian of the potential. Thus the time scale of interest in particle beams is the effective betatron period, in plasmas it is the plasma period, in gravitational systems it is the free-fall time, etc.

One must take care to consider mixing time within the framework of *globally* chaotic behavior. An initially localized perturbation will have "mixed" only after it has grown to the point that its scale is comparable to the size of the system. By contrast, discreteness noise associated with single-particle interactions can also generate rapid exponential growth of perturbations, but the growth saturates on small scales and is therefore not representative of mixing [15].

The foregoing treatment has the attractive feature of being anchored to the behavior of a generic, *i.e.*, "average" trajectory that samples the global phase space established by a coarse-grained potential. By design it appropriately predicts zero mixing due to local interparticle interactions. Concomitantly, it predicts infinite chaotic mixing time for linear space-charge forces. The particle density is then uniform; Eqs. (9) and (11) make clear that the corresponding Lyapunov exponent is zero. The same is true, of course, for linear external forces and zero space charge.

With a more general metric, specifically a Finsler metric, geometrodynamics can incorporate a potential that is both time-dependent and velocity-dependent [16]. It is thereby applicable to a wide variety of beam-dynamics problems. For example, one might be able to devise a geometric measure of chaos over a Finsler manifold for use in rapidly mapping the features of a system's Poincaré surface of section, as has been done for the Hénon-Heiles potential [17]. If so, then the method would provide efficient calculation of dynamic apertures in circular machines in which magnet nonlinearities, and perhaps beam-beam interactions, establish parametric resonances that gradually degrade the beam. Traditional methods require very long integration times and their attendant numerical difficulties.

For a time-dependent potential, one can postulate a lowest-order approximation in which the principal effect of the time dependence on mixing is to establish additional parametric resonances. They could arise, for example, by way of resonant coupling between space-charge modes and periodicities in the transport lattice. A rough estimate of the mixing time would then follow from Eq. (11), with the time-averaged potential defining the micro-canonical ensemble. Insofar as the result involves the depressed betatron frequency, it agrees with "relaxation" times seen in beams that are mismatched to their transport lattices [13]. However, one must not infer too much from this conjecture. The Eisenhart metric, being independent of velocity space, cannot account for energy exchange between space-charge modes and individual particles. Thus, important processes such as violent relaxation and attendant halo formation [18] are missing. In principle, they are accessible with a Finsler metric.

It will interesting to compare geometrodynamic predictions against numerical experiments that specifically concern the evolution of particle ensembles. The fate of initially localized perturbations ostensibly arising in response to transient forces external to the bunch is of special interest. Inasmuch as the formalism yields mixing time versus total particle energy, it provides a "diagnostic" of chaotic behavior. Particles passing through regions with strongly nonlinear forces will tend to have shorter mixing times than those that do not.

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FIGURES

FIG. 1. Lyapunov exponent λ in units of $\sqrt{\kappa}$ plotted as a function of $\rho = \sigma/\kappa$; the quantities κ and σ are defined in the text.